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1982 J. Phys. A: Math. Gen. 15 L97

(http://iopscience.iop.org/0305-4470/15/3/003)

J. Phys. A: Math. Gen. **15** (1982) L97–L101. Printed in Great Britain

LETTER TO THE EDITOR

Definite integrals as solutions for the $x^2+\lambda x^2/(1+gx^2)$ potential

George P Flessas
Department of Natural Philosophy, University of Glasgow, Glasgow G12 8QQ, Scotland

Received 20 October 1981

Abstract. We present exact solutions and eigenvalues for the Schrödinger equation with the $x^2+\lambda x^2/(1+gx^2)$ interaction on the half axis $x \geqslant 0$. The solutions are given in the form of definite integrals and the eigenvalues by means of a well defined limiting procedure.

We consider the Schrödinger equation

$$y''(x)+[E-x^2-\lambda x^2/(1+gx^2)]y(x)=0 \qquad 0\leqslant x\leqslant\infty,\ g>0 \tag{1}$$

which appears in various fields as summarised by Bessis and Bessis (1980) and Mitra (1978). Our aim is to continue the investigation started in an earlier paper (Flessas 1981) and find further rigorous eigenfunctions and eigenvalues E for equation (1). Using

$$x^2=t \qquad y(x)=\exp(-\tfrac{1}{2}t)h(t) \tag{2}$$

we deduce from equation (1)

$$4t(1+gt)h''(t)+[-4gt^2+2t(g-2)+2]h'(t)+[E-1+t(Eg-\lambda-g)]h(t)=0. \tag{3}$$

In contrast to Flessas (1981) where polynomial-type solutions for equation (3) were found here we make the ansatz

$$h(t)=\int_0^\infty \exp(-\tfrac{1}{2}zt)f(z)\,\mathrm{d}z \qquad t>0. \tag{4}$$

Under the assumption, which is going to be justified later, that the integral in equation (4) exists we can build $h'(t)$, $h''(t)$ and insert them into equation (3). After some algebra we obtain the following equations that have yet to be fulfilled:

$$z(z+2)f''(z)+\left[\frac{z^2}{2g}+z\left(\frac{7g+2}{2g}\right)+\frac{E+7}{2}-\frac{\lambda}{2g}\right]f'(z)+\left(\frac{3z+6g+3+E}{4g}\right)f(z)=0 \tag{5}$$

$$F(\infty)-F(0)=0$$

$$F(z)=\exp(-\tfrac{1}{2}tz)[2z^2f(z)+2tgz^2f(z)+6gzf(z)+4gz^2f'(z) \tag{6}$$
$$+4gztf(z)+8gzf'(z)+4zf(z)+2(Eg-\lambda)f(z)+6gf(z)].$$

Equation (6) comes from the integrated part of the integrals inserted into equation (3).

Our next task consists in satisfying equations (5)–(6). We consider first equation (5). Such a type of differential equation, which also appears in the calculation of the possible

0305-4470/82/030097+05\$02.00

energies of the ion of molecular hydrogen in the framework of Schrödinger's theory, has been investigated by Wilson (1928). Equation (5), however, is somewhat more general than that of Wilson (1928). Nevertheless we can use some of the results of that work by imposing, as follows by comparing the two differential equations in question, on E, λ, g the condition

$$Eg = \lambda. \tag{7}$$

Then the solution of equation (5) reads, as can be easily verified,

$$f(z) = \sum_{m=0}^{\infty} c_m (z+2)^{-m-3/2} \tag{8}$$

$$a_m c_m + b_{m-1} c_{m-1} + d_{m-2} c_{m-2} = 0 \qquad m \geq 0,\ c_{-1} = c_{-2} = 0,\ c_0 = 1$$

$$a_m = -m/2g,\ b_{m-1} = m(m - \tfrac{3}{2} + g^{-1}) + \frac{E - 1 + 2g}{4g} \qquad d_{m-2} = -2(m - \tfrac{1}{2})(m - \tfrac{5}{4}). \tag{9}$$

The power series in equation (8) converges for $0 < z \leq \infty$. The second linearly independent solution is physically unacceptable on the grounds of its behaviour for $z \to \infty$. Further it can be shown (Wilson 1928), as it is usually done by studying c_m/c_{m-1} for $m \to \infty$, that for $z \to 0$, $f(z)$ becomes

$$f(z) = A(z+2)^{-3/4} z^{-3/4} \qquad A = A(E, g),\ z \to 0 \tag{10}$$

and also $f(z) \xrightarrow[z\to\infty]{} 0$. Hence the existence of the integral in equation (4) is ensured. Now for the investigation of equation (6) we shall need the $\lim_{z\to 0} f'(z)$. To this end we note that the differentiation of equation (8) for $z \in (0, \infty]$ is permitted:

$$f'(z) = -\frac{3f(z)}{2(z+2)} - \frac{1}{(z+2)^{5/2}} \sum_{m=0}^{\infty} c'_m (z+2)^{-m} \qquad c'_m = mc_m,\ 0 < z \leq \infty. \tag{11}$$

To find the behaviour of the sum in equation (11) for $z \to 0$ we remark that by virtue of equation (9) it can be seen (Wilson 1928) that $(c_m/c_{m-1}) \to 2$ as $m \to \infty$. We set $c'_m/c'_{m-1} = 2(1 - u/m)$ for m very large, and by utilising equation (9) we get by equating powers of m that $u = -\frac{3}{4}$. Therefore for large m values c'_m approximates the corresponding coefficient of w^m in the binomial series

$$(1-2w)^{-7/4} = 1 + \sum_{m=1}^{\infty} \frac{(-\frac{7}{4})(1-\frac{7}{4}) \ldots [-(m-1)-\frac{7}{4}]}{m!} (-2w)^m \qquad w = \frac{1}{z+2}$$

and consequently the sum in equation (11) becomes for $z \to 0$ equal to the expression $B(z+2)^{7/4} z^{-7/4}$, $B = B(E, g)$. So from equation (11) we obtain

$$f'(z) = -\tfrac{3}{2} A(z+2)^{-7/4} z^{-3/4} - B(z+2)^{-3/4} z^{-7/4} \qquad z \to 0. \tag{12}$$

We examine now equation (6). Owing to equations (7), (10) and (12) we observe that condition (6) holds provided

$$6gf(z) + 8gzf'(z) = 0 \qquad z \to 0$$

or, equivalently,

$$A - 4B/3 = 0. \tag{13}$$

To fulfil equation (13) it is sufficient to require

$$A=1 \qquad B=\tfrac{3}{4}. \tag{14}$$

Equation (14) implies that $f(z)(z+2)^{3/2}$ and the sum in equation (11) become, respectively, for $z\to 0$ identical with $(1-2w)^{-3/4}$ (cf equation (10)) and $3(1-2w)^{-7/4}/4$. In other words c_m and c'_m satisfy

$$c_m=\frac{(-\frac{3}{4})(-1-\frac{3}{4})\ldots[-(m-1)-\frac{3}{4}]}{m!}(-2)^m \qquad m\to\infty$$

$$c'_m=\frac{(\frac{3}{4})(-\frac{7}{4})(-1-\frac{7}{4})\ldots[-(m-1)-\frac{7}{4}]}{m!}(-2)^m \qquad m\to\infty. \tag{15}$$

Using the standard formula from the gamma function theory $\Gamma(a+m)=a(a+1)\ldots(a+m-1)\Gamma(a)$ it can be verified that, as expected, the two relations in equation (15) are the same. Thus we need only consider one of them, say the second one. Then by writing down first the closed form for c'_m, which follows from equation (9),

$$c'_m=\frac{m(-2)^m g^m D_m}{m!} \qquad m\geqslant 1$$

$$D_m=\begin{vmatrix} -b_0 & a_1d_0 & 0 & 0 & 0 & \ldots & 0\\ 1 & -b_1 & -a_2 & 0 & 0 & \ldots & 0\\ 0 & -d_1 & -b_2 & a_3 & 0 & \ldots & .\\ 0 & 0 & d_2 & -b_3 & -a_4 & . & .\\ . & . & . & . & . & \ddots & .\\ . & . & . & & & -b_{m-2} & (-1)^m a_{m-1}\\ . & . & . & & & (-1)^m d_{m-2} & -b_{m-1} \end{vmatrix} \qquad m\geqslant 3 \tag{16}$$

$$D_m=-b_{m-1}D_{m-1}-d_{m-2}a_{m-1}D_{m-2},\ D_1=-b_0,\ D_2=b_0b_1-a_1d_0 \qquad m\geqslant 3$$

we deduce for the second of the conditions (15)

$$mg^mD_m=(\tfrac{3}{4})(-\tfrac{7}{4})(-1-\tfrac{7}{4})\ldots[-(m-1)-\tfrac{7}{4}] \qquad m\to\infty. \tag{17}$$

Condition (17) is exact. To determine the energy eigenvalues $E(g)$ from it we have to use the asymptotic expressions for D_m and $(-\frac{7}{4})\ldots[-(m-1)-\frac{7}{4}]$. Then m will cancel, as otherwise c'_m would not approximate the corresponding coefficient in $(1-2w)^{-7/4}$ as shown prior to equation (12). As a consequence we shall obtain an exact relation $R(E, g)=0$ from which $E(g)$ can be calculated. We note, however, that both sides of equation (17) are very rapidly increasing functions of m (they increase practically stronger than $m!$). Hence for moderate m values they will approximately attain their asymptotic forms and as a result we can obtain a condition connecting E and g which is an approximation to the exact $R(E, g)=0$. It is straightforward to demonstrate the foregoing for somewhat large g values. We choose $g=100$. Then for $m=2, 3, 4$ equation (17) gives, respectively, the following approximations to the exact E: $E^{(2)}=-610$, $E^{(3)}=-668$, $E^{(4)}=-620$. These numbers suggest a relatively rapid convergence of the procedure with the energy lying between -610 and -668. The corresponding λ is found from equation (7). As in Flessas (1981) our method includes $\lambda<0$, whereas the numerical approaches of Bessis and Bessis (1980) and Mitra (1978) are readily applicable only for $\lambda+1>0$.

We have yet to accommodate for the case $t=0$ which was excluded in equation (4). This was so because if $t=0$ the term $z^2 f(z)$ in equation (6), $f(z)$ being given by equation (8), causes $F(\infty)\to\infty$. Thus although the integral in equation (4) still exists for $t=0$, it is not a solution of equation (3) for $t=0$. To include now $t=0$ in our method we have to modify slightly the potential $V(x)=x^2+\lambda x^2/(1+gx^2)$ in equation (1) without changing the physically important eigenvalue E as calculated above. This can be done by considering the modified interaction

$$V_1(x)=\begin{cases} V(x) & x>\varepsilon \\ 0 & 0\leqslant x\leqslant\varepsilon \end{cases} \tag{18}$$

ε being an arbitrarily small positive number. Then for $0\leqslant x\leqslant\varepsilon$ $y(x)=\alpha\exp(\mathrm{i}E^{1/2}x)+\beta\exp(-\mathrm{i}E^{1/2}x)$, where E is given by equation (17) and α, β are determined from the usual continuity conditions at $x=\varepsilon$. Since $V(x)$ is a well behaved function for $x\to 0$, we can choose the $\varepsilon>0$ as small as we like without altering the physical situation which corresponds to keeping the $V(x)$ for $x\geqslant 0$. As a consequence we have the eigenparameter E determined by equation (17) for all $x\geqslant 0$ and $y(x)$ defined by equations (2)–(4) valid for all $x>0$.

We recall that the series defining $f(z)$ in equation (8) diverges for $z=0$ and therefore, although the integrals appearing after inserting equation (8) into equation (4) are known (Gradshteyn and Ryzhik 1965), one must examine the convergence of the integration result for $0<t\leqslant\infty$. We obtain

$$h(t)=\exp(t)(\tfrac{1}{2}t)^{1/2}\sum_{m=0}^{\infty}c_m\Gamma(-m-\tfrac{1}{2},t)\frac{t^m}{2^m} \tag{19}$$

where $\Gamma(u,v)$ is the incomplete gamma function. By applying the standard criterion for the convergence of series we readily see that the series in equation (19) diverges for all $t>0$. This shows that term-by-term integration in equation (4) is not permitted.

The difference between the present work and that of Flessas (1981) lies in the following. The structure of equation (5) reveals that it cannot possess polynomial-like solutions (cf equation (9) where $d_i\neq 0$). Thus $h(t)$ in equation (4) is also non-polynomial. Nevertheless, in the harmonic limit $g\to 0$ equation (2) should reduce to some eigenfunction of the harmonic oscillator in order that the procedure proposed is consistent. Hence we conclude that the only possibility remaining is $h(t)=0$ for all t, which implies that equation (5) for $g=0$ should give $f(z)=0$ for all z as the solution compatible with the existence of the integral in equation (4). Taking into account that for $g=0$ and in order to obtain the harmonic oscillator equation from equation (1) the energy has to be positive, it can be easily seen that equation (5) does indeed give $f(z)=0$ for all z, the other solution being discarded since it makes the integral in equation (4) divergent at $z=0$. This situation suggests that in any attempt to obtain the solutions (4) by means of some expansion in harmonic oscillator functions all the expansion coefficients vanish for $g=0$. Therefore we have the previously announced difference with Flessas (1981) where the exact solutions of equation (1), as they are finite linear combinations of oscillator eigenfunctions, reduce for $g=0$ to an eigenfunction of the harmonic oscillator.

To sum up we have given exact solutions and eigenvalues for equation (1) and demonstrated their applicability. Comparison with relevant previous works has been done and the differences have been pointed out.

I wish to thank D Sutherland and R R Whitehead for valuable comments.

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